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Uniform product of $A_{g,n}(V)$ for an orbifold model V and G-twisted Zhu algebra

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Abstract

Let V be a vertex operator algebra and G a finite automorphism group of V. For each $g \in G$ and nonnegative rational number $n \in \mathbb{Z}/|g|$, an associative algebra $A_{g,n}(V)$ plays an important role in the theory of vertex operator algebras, but the given product in $A_{g,n}(V)$ depends on the eigenspaces of g. We show that if V has no negative weights then there is a uniform definition of products on V and we introduce a G-twisted Zhu algebra $A_{G,n}(V)$ which covers all $A_{g,n}(V)$.

Let V be a simple vertex operator algebra with no negative weights and let $\mathcal{S}$ be a finite set of inequivalent irreducible twisted V-modules which is closed under the action of G. There is a finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G, \mathcal{S})$ for a suitable 2-cocycle naturally determined by the G-action on $\mathcal{S}$. We show that a duality theorem of Schur–Weyl type holds for the actions of $\mathcal{A}_\alpha(G, \mathcal{S})$ and V^G on the direct sum of twisted V-modules in $\mathcal{S}$ as an application of the theory of $A_{G,n}(V)$. It follows as a natural consequence of the result that for any $g \in G$ every irreducible g-twisted V-module is a completely reducible V^G-module.

1. Introduction

Let V be a vertex operator algebra (cf. [1,9,10]) and G a finite automorphism group of V order T. For an investigation of V-modules, a Zhu algebra $A(V) = V/O(V)$ (or its extension $A_n(V) = V/O_n(V)$), which was introduced in [12] and [5], plays an important

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role. For a g-twisted V-module, a series of associative algebras $A_{g,n}(V)=V/O_{g,n}(V)$ was introduced in [6] for nonnegative $n\in\mathbb{Z}/T$ and plays a similar role. However, the definition of product $u*_{g,n}v$ in $A_{g,n}(V)$ depends on the choice of eigenspaces of g containing u. Namely, for $g\in G$, V decomposes into the direct sum of eigenspaces $V=\bigoplus_{r=0}^{T-1}V^{(g,r)}$, where $V^{(g,r)}$ is the eigenspace of g with eigenvalue $e^{-2\pi\sqrt{-1}r/T}$. If $u\in V^{(g,0)}$, then

$$u*_{g,n}v\equiv\sum_{m=0}^{l}(-1)^m\binom{m+h}{l}\operatorname{Res}_z Y(u,z)v\frac{(1+z)^{\mathrm{wt}(u)+l}}{z^{l+m+1}}\quad\bigl(\operatorname{mod}O_{g,n}(V)\bigr)$$

and if $u\in V^{(g,r)}$ $(r\neq 0)$, then $u*_{g,n}v\equiv 0\ (\operatorname{mod}O_{g,n}(V))$, where $n=l+i/T$ with $0\leqslant i\leqslant T-1$ and $l\in\mathbb{Z}_{\geqslant 0}$. One of the purposes in this paper is to show that if V has no negative weights then for $u,v\in V$, there is a unified form $*_n$ of products which does only depend on the weights of u and v such that

$$u*_n v\equiv u*_{g,n}v\quad\bigl(\operatorname{mod}O_{g,n}(V)\bigr)$$

for any $g\in G$. Using this product, a natural map

$$\phi_n:V\to\bigoplus_{g\in G}A_{g,n}(V)$$

given by $\phi_n(u)=(u,u,\dots,u)$ becomes an algebra homomorphism:

$$\phi_n(u*_n v)=\bigl(u*_{g,n}v+O_{g,n}(V)\bigr)_{g\in G}.$$

Set $A_{G,n}(V)=V/\bigcap_{g\in G}O_{g,n}(V)$. We will call $(A_{G,n}(V),*_n)$ a G-twisted Zhu algebra. We will show that if V is simple, then $A_{G,n}(V)\simeq\bigoplus_{g\in G}A_{g,n}(V)$ as algebras.

We will apply the theory of $A_{G,n}(V)$ to a problem of the representation theory of V^G. One of the important problems in the orbifold conformal field theory is to determine the V^G-modules. In particular, it is one of the main conjectures that if V is rational, then V^G is rational, that is, every V^G-module is completely reducible (cf. [2]). For a simple VOA V, it is shown in [8] that every irreducible V-module is a completely reducible V^G-module as a natural consequence of a duality theorem of Schur–Weyl type. Another important category is a twisted module (cf. [4]). Namely, for any $g\in G$, any g-twisted module is also a V^G-module. The same results as those in the untwisted case are shown in [11] for irreducible g-twisted V-modules when g is in the center of G.

In this paper we extend their results to any $g\in G$ and any irreducible g-twisted V-module. Let's state our results more explicitly. Let V be a simple vertex operator algebra with no negative weights. There is a natural right G-action on the set of all inequivalent irreducible twisted V-modules. Let $\mathcal{S}$ be a finite set of inequivalent irreducible twisted V-modules which is closed under the action of G. We define a finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G,\mathcal{S})$ over $\mathbb{C}$ associated to G, $\mathcal{S}$ and a suitable 2-cocycle α naturally determined by the G-action on $\mathcal{S}$. The algebra $\mathcal{A}_\alpha(G,\mathcal{S})$

is constructed in more general setting in [8] and is called a generalized twisted double. We show a duality theorem of Schur–Weyl type for the actions of V^G and $\mathcal{A}_\alpha(G,\mathcal{S})$ on the direct sum of twisted V-modules in $\mathcal{S}$ which is denoted by $\mathcal{M}$. That is, each simple $\mathcal{A}_\alpha(G,\mathcal{S})$ occurs in $\mathcal{M}$ and its multiplicity space is an irreducible V^G-module. Moreover, the different multiplicity spaces are inequivalent V^G-modules. It follows as a natural consequence of the result that for any $g \in G$ every irreducible g-twisted V-module is a completely reducible V^G-module. The theory of $A_{G,n}(V)$ allows us to reduce an infinite dimensional problem to a finite dimensional one.

This paper is organized as follows. In Section 2, we first review a series of associative algebras $A_{g,n}(V)$ of a vertex operator algebra constructed in [6]. We introduce $(A_{G,n}(V), *_n)$ and study their properties. In Section 3, we define a finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G,\mathcal{S})$ over $\mathbb{C}$ and show a duality theorem of Schur–Weyl type as an application of the theory of $A_{G,n}(V)$. In Section 4, we compute the determinant of a matrix used in Section 2.

2. Existence of unified form of product and G-twisted Zhu algebras

We fix some notation which will be in force throughout the paper. V is a vertex operator algebra with no negative weights: $V = \bigoplus_{n=0}^{\infty} V_n$. G is a finite automorphism group of V of order T. For $g \in G$, set $V^{(g,r)} = \{u \in V \mid gu = e^{-2\pi\sqrt{-1}r/T}u\}$ for $0 \leqslant r \leqslant T-1$. For $u \in V$ we denote by $u^{(g,r)}$ the rth component of u in the decomposition $V = \bigoplus_{r=0}^{T-1} V^{(g,r)}$, that is, $u = \sum_{r=0}^{T-1} u^{(g,r)}$, $u^{(g,r)} \in V^{(g,r)}$ $(0 \leqslant r \leqslant T-1)$.

In this section, our main purpose is to show that there is a unified product form $u *_n v$ which depends only on the weights of u and v. We introduce an associative algebra $(A_{G,n}(V), *_n)$ of V associated to G and nonnegative $n \in \mathbb{Z}/T$ and study their properties.

We first recall the associative algebra $A_{g,n}(V)$ of V associated with $g \in G$ and nonnegative $n \in \mathbb{Z}/T$ introduced in [6]. This algebra was first introduced in [12] for the case when g is the identity element and $n = 0$. Fix nonnegative $n = l + i/T \in \mathbb{Z}/T$ with l a nonnegative integer and $0 \leqslant i \leqslant T-1$.

For $0 \leqslant r \leqslant T-1$ we define $\delta_i(r) = 1$ if $i \geqslant r$ and $\delta_i(r) = 0$ if $i < r$. We also set $\delta_i(T) = 1$. Let $g \in G$. Let $O_{g,n}(V)$ be the linear span of all $u \circ_{g,n} v$ and $(L(-1)+L(0))v$ where for homogeneous $u \in V$ and $v \in V$, $u \circ_{g,n} v$ denotes

$$\sum_{r=0}^{T-1} \operatorname{Res}_z Y\bigl(u^{(g,r)}, z\bigr)v \frac{(1+z)^{\mathrm{wt}(u)-1+\delta_i(r)+l+r/T}}{z^{2l+\delta_i(r)+\delta_i(T-r)}}.$$

Define the linear space $A_{g,n}(V)$ to be the quotient $V/O_{g,n}(V)$. We also define a second product $*_{g,n}$ on V for homogeneous $u \in V$ and $v \in V$ by

$$u *_{g,n} v = \sum_{m=0}^{l} (-1)^m \binom{m+l}{l} \operatorname{Res}_z Y\bigl(u^{(g,0)}, z\bigr)v \frac{(1+z)^{\mathrm{wt}(u)+l}}{z^{l+m+1}}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^{l-j} \binom{2l-j}{l} \binom{\mathrm{wt}(u)+l}{k-j} u^{(g,0)}_{-2l-1+k} v.$$

Extend this linearly to obtain a bilinear product on V. Note that if $i = j + kT/|g|$ where $0 \leqslant j \leqslant T/|g| - 1$ and $0 \leqslant k \leqslant |g| - 1$, then $A_{g,n}(V) = A_{g,l+k/|g|}(V)$. We recall the following properties of $O_{g,n}(V)$ and $A_{g,n}(V)$.

Lemma 1 [6, Lemmas 2.1, 2.2, Theorem 2.4, and Proposition 2.5].

(1) $\bigoplus_{r=1}^{T-1} V^{(g,r)} \subset O_{g,n}(V)$.
(2) *For homogeneous $u \in V$, $v \in V$ and integers $0 \leqslant k \leqslant m$,*

$$\sum_{r=0}^{T-1} \mathrm{Res}_z\, Y\big(u^{(g,r)}, z\big) v \frac{(1+z)^{\mathrm{wt}(u)-1+\delta_i(r)+l+r/T+k}}{z^{2l+\delta_i(r)+\delta_i(T-r)+m}} \in O_{g,n}(V).$$

(3) $(A_{g,n}(V), *_{g,n})$ *is an associative algebra with the identity element.*
(4) *The identity map on V induces an onto algebra homomorphism from $A_{g,n}(V)$ to $A_{g,n-1/T}(V)$.*

Set $A_{G,n}(V) = V/\bigcap_{g\in G} O_{g,n}(V)$. We state our main result in this section.

Theorem 1.

(1) *For $u, v \in V$, there exists a unified product $u *_n v$ for $A_{g,n}(V)$ for any $g \in G$. That is,*

$$u *_n v \equiv u *_{g,n} v \quad \big(\mathrm{mod}\, O_{g,n}(V)\big)$$

for any $g \in G$. In particular, $(A_{G,n}(V), *_n)$ is an associative algebra. Moreover, if V is simple, then the injective map $\varphi_n : A_{G,n}(V) \to \bigoplus_{g\in G} A_{g,n}(V)$ defined by

$$\varphi_n\Big(u + \bigcap_{g\in G} O_{g,n}(V)\Big) = \sum_{g\in G} \big(u + O_{g,n}(V)\big) \quad \text{for } u \in V$$

is an onto algebra homomorphism. That is, $A_{G,n}(V) \simeq \bigoplus_{g\in G} A_{g,n}(V)$ as algebras.
(2) *The identity map on V induces an onto algebra homomorphism from $A_{G,n}(V)$ to $A_{G,n-1/T}(V)$.*

Remark. We will construct $u *_n v$ as a linear combination of $\{u_j v \mid j \in \mathbb{Z}\}$ in the proof of Theorem 1: $u *_n v = \sum_{j\in\mathbb{Z}} \gamma_j u_j v$ where $\gamma_j \in \mathbb{C}$ ($j \in \mathbb{Z}$) are all zero except finite numbers. Then the set $\{\gamma_j\}_{j\in\mathbb{Z}}$ depends only on n and the weights of u and v.

We will call $A_{G,n}(V)$ a G-twisted Zhu algebra. We need some lemmas in order to show Theorem 1. Let $u, v \in V$ and fix $M \in \mathbb{Z}$ with $M \geqslant \mathrm{wt}(u) + \mathrm{wt}(v) - 1$ in the next three lemmas. Note that $u_j^{(g,r)}v=0$ for all $g\in G, 0\leqslant r\leqslant T-1$, and $j>M$. Set $Q_r=\mathrm{wt}(u)-1+\delta_i(r)+l+r/T$ for $0\leqslant r\leqslant T-1$.

Lemma 2. *For $q\in\mathbb{Z}$ with $-2l-1\geqslant q$ and $m=1,2,\ldots,$ we have*

$$\begin{aligned}u_{q-m}v\equiv{}&\sum_{k=0}^{Q_0-1}\sum_{j=0}^{k}\binom{Q_0}{j}\binom{-Q_0}{k+m-j}u_{q+k}^{(g,0)}v\\&+\sum_{r=1}^{T-1}\sum_{k=0}^{M-q}\sum_{j=0}^{k}\binom{Q_r}{j}\binom{-Q_r}{k+m-j}u_{q+k}^{(g,r)}v\quad\bigl(\mathrm{mod}\,O_{g,n}(V)\bigr)\end{aligned}$$

for any $g\in G$.

Proof. Let $g\in G$. We may assume that u is homogeneous and $u\in V^{(g,r)}$ for $r\in\{0,\ldots,T-1\}$. Since $2\geqslant\delta_i(r)+\delta_i(T-r)$, we have

$$O_{g,n}(V)\ni\mathrm{Res}_z\,Y(u,z)v\frac{(1+z)^{Q_r}}{z^{2l+2+m}}=\sum_{j=0}^{\infty}\binom{Q_r}{j}u_{-2l-2-m+j}v$$

for any nonnegative integer m from Lemma 1(2). Let $q\in\mathbb{Z}$ with $q\leqslant-2l-1$. Since $q\leqslant-2l-1$, we have

$$O_{g,n}(V)\ni\sum_{j=0}^{\infty}\binom{Q_r}{j}u_{q-m+j}v=\sum_{j=0}^{m-1}\binom{Q_r}{j}u_{q-m+j}v+\sum_{k=0}^{\infty}\binom{Q_r}{m+k}u_{q+k}v$$

for any positive integer m. Namely, we have

$$\begin{aligned}u_{q-1}v&\equiv-\binom{Q_r}{1}u_qv-\binom{Q_r}{2}u_{q+1}v-\cdots,\\u_{q-2}v+\binom{Q_r}{1}u_{q-1}v&\equiv-\binom{Q_r}{2}u_qv-\binom{Q_r}{3}u_{q+1}v-\cdots,\\u_{q-3}v+\binom{Q_r}{1}u_{q-2}v+\binom{Q_r}{2}u_{q-1}v&\equiv-\binom{Q_r}{3}u_qv-\binom{Q_r}{4}u_{q+1}v-\cdots,\\&\ \,\vdots\qquad\qquad\qquad\vdots\end{aligned}\tag{2.1}$$

module $O_{g,n}(V)$. Solving these congruent equations, $u_{q-m}v$ is congruent to a linear combination of $\{u_qv,u_{q+1}v,\ldots,u_Mv\}$ modulo $O_{g,n}(V)$ for any positive integer m. If $r=0$, then $Q_r=\mathrm{wt}(u)+l\in\mathbb{Z}_{\geqslant0}$ and so $u_{q-m}v$ is congruent to a linear combination of $\{u_qv,u_{q+1}v,\ldots,u_{q+Q_0-1}v\}$.

Replacing $u_j v$ by x^j, we will investigate the coefficients by viewing the above equations as a Lorentz series. Set

$$f_r(x)=\sum_{j=0}^{\infty}\binom{Q_r}{j}x^j. \tag{2.2}$$

We note that if $\sum_{j\in\mathbb{Z}}\alpha_j x^j\in\mathbb{C}[x^{-1}]x^{-2l-2}f_r(x)$, then $\sum_{j\in\mathbb{Z}}\alpha_j u_j v\in O_{g,n}(V)$ and $f_r(x)$ depends only on $N=\mathrm{wt}(u)$, r and $n=l+i/T$. Set $O^r(N:x)=\mathbb{C}[x^{-1}]x^{-2l-2}f_r(x)\subset\mathbb{C}[x^{-1}][\![x]\!]$.

Let m be a positive integer. In order to express $u_{q-m}v$ by a linear combination of $\{u_q v,u_{q+1}v,\dots,u_M v\}$ modulo $O_{g,n}(V)$, we first write $1/f_r(x)$ as a sum

$$\frac{1}{f_r(x)}=\bigl(f_r(x)^{-1}\bigr)_{<m}\bigl(f_r(x)^{-1}\bigr)_{\geqslant m},$$

where $(f_r(x)^{-1})_{<m}$ is the part whose terms have degree less than m and $(f_r(x)^{-1})_{\geqslant m}$ is the part whose terms have degree greater than or equal to m. In particular, we have $(f_r(x)^{-1})_{\geqslant m}\in x^m\mathbb{C}[\![x]\!]$. Since $(x^{q-1}f_r(x))(f_r(x)^{-1})_{<m}/x^{m-1}\equiv 0\ (\mathrm{mod}\,O^r(N:x))$, we obtain

$$\begin{aligned}
x^{q-m}&\equiv x^{q-m}-x^{q-1}f_r(x)\frac{(f_r(x)^{-1})_{<m}}{x^{m-1}}\\
&\equiv x^{q-m}-x^{q-1}f_r(x)\frac{f_r(x)^{-1}-(f_r(x)^{-1})_{\geqslant m}}{x^{m-1}}\\
&\equiv x^{q-m}f_r(x)\bigl(f_r(x)^{-1}\bigr)_{\geqslant m}\\
&\equiv x^{q-m}\left(\sum_{j=0}^{\infty}\binom{Q_r}{j}x^j\right)\left(\sum_{k=m}^{\infty}\binom{-Q_r}{k}x^k\right)\\
&\equiv\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{Q_r}{j}\binom{-Q_r}{k+m-j}x^{q+k}\quad\bigl(\mathrm{mod}\,O^r(N:x)\bigr).
\end{aligned}$$

Hence we have

$$\begin{aligned}
u_{q-m}v&\equiv\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{Q_r}{j}\binom{-Q_r}{k+m-j}u_{q+k}v\\
&\equiv\sum_{k=0}^{M-q}\sum_{j=0}^{k}\binom{Q_r}{j}\binom{-Q_r}{k+m-j}u_{q+k}v\quad\bigl(\mathrm{mod}\,O_{g,n}(V)\bigr).\qquad\square
\end{aligned}$$

Lemma 3. *For $r\in\{0,\dots,T-1\}$ and $p\in\mathbb{Z}$, there exists $F^{r,p}(u,v,n)\in V$ such that*

$$F^{r,p}(u,v,n) \equiv u_p^{(g,r)} v \quad \bigl(\operatorname{mod} O_{g,n}(V)\bigr) \tag{2.3}$$

for any $g \in G$.

Proof. We may assume $p \leqslant M$. Define a set

$$S = \left\{ (s,k) \;\middle|\; \begin{array}{ll} s = 0,1,\dots,T-1, & \\ k = 0,\dots,Q_0-1 & \text{for } s=0, \text{ and} \\ k = 0,\dots,M-q & \text{for } s = 1,\dots,T-1 \end{array} \right\}.$$

Let $q = \min\{p, -2l-1\}$ and set

$$F^{r,p}(u,v,n) = \begin{cases} u_p v - \sum_{m=1}^{|S|} \lambda(0)_{q-m} u_{q-m} v & \text{if } r = 0, \\ \sum_{m=1}^{|S|} \lambda(r)_{q-m} u_{q-m} v & \text{if } 1 \leqslant r \leqslant T-1, \end{cases}$$

where $\lambda(r)_{q-m} \in \mathbb{C}$ $(m = 1,2,\dots,|S|)$. We show that it is possible to choose a set $\{\lambda(r)_{q-m}\}_{m=1}^{|S|}$ which depends only on $\operatorname{wt}(u), \operatorname{wt}(v)$ and n such that the assertion holds. For $g \in G$, we have

$$\begin{aligned} &\sum_{m=1}^{|S|} \lambda(r)_{q-m} u_{q-m} v \\ &\quad \equiv \sum_{m=1}^{|S|} \lambda(r)_{q-m} \left(\sum_{k=0}^{Q_0-1} \sum_{j=0}^{k} \binom{Q_0}{j} \binom{-Q_0}{k+m-j} u_{q+k}^{(g,0)} v \right. \\ &\qquad \left. + \sum_{s=1}^{T-1} \sum_{k=0}^{M-q} \sum_{j=0}^{k} \binom{Q_s}{j} \binom{-Q_s}{k+m-j} u_{q+k}^{(g,s)} v \right) \\ &\quad \equiv \sum_{k=0}^{Q_0-1} \sum_{m=1}^{|S|} \lambda(r)_{q-m} \sum_{j=0}^{k} \binom{Q_0}{j} \binom{-Q_0}{k+m-j} u_{q+k}^{(g,0)} v \\ &\qquad + \sum_{s=1}^{T-1} \sum_{k=0}^{M-q} \sum_{m=1}^{|S|} \lambda(r)_{q-m} \sum_{j=0}^{k} \binom{Q_s}{j} \binom{-Q_s}{k+m-j} u_{q+k}^{(g,s)} v \quad \bigl(\operatorname{mod} O_{g,n}(V)\bigr) \end{aligned}$$

from Lemma 2. Comparing both sides of formula (2.3), we have $|S|$ linear equations:

- In the case $1 \leqslant r \leqslant T-1$,

$$\sum_{m=1}^{|S|} \lambda(r)_{q-m} \sum_{j=0}^{k} \binom{Q_s}{j} \binom{-Q_s}{k+m-j} = \begin{cases} 1 & \text{if } s = r \text{ and } q+k = p, \\ 0 & \text{otherwise.} \end{cases}$$

- In the case $r=0$,

$$\sum_{m=1}^{|S|} \lambda(0)_{q-m} \sum_{j=0}^{k} \binom{Q_s}{j} \binom{-Q_s}{k+m-j} = \begin{cases} 1 & \text{if } s \neq 0 \text{ and } q+k=p, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Here (s,k) ranges over S. We denote $\sum_{j=0}^{k} \binom{Q_s}{j}\binom{-Q_s}{k+m-j}$ by $\alpha_m^{s,k}$ for $(s,k) \in S$ and a positive integer m. We set a $(|S| \times |S|)$-matrix

$$A_1 = \begin{bmatrix} \alpha_1^{0,0} & \cdots & \alpha_1^{0,Q_0-1} & \alpha_1^{1,0} & \cdots & \cdots & \alpha_1^{T-1,M-p} \\ \alpha_2^{0,0} & \cdots & \alpha_2^{0,Q_0-1} & \alpha_2^{1,0} & \cdots & \cdots & \alpha_1^{T-1,M-p} \\ \vdots & & \vdots & \vdots & & & \vdots \\ \alpha_{|S|}^{0,0} & \cdots & \alpha_{|S|}^{0,Q_0-1} & \alpha_{|S|}^{1,0} & \cdots & \cdots & \alpha_{|S|}^{T-1,M-p} \end{bmatrix}.$$

It is sufficient to show that the matrix A_1 is non-singular in order to prove Eqs. (2.3) have a solution $\{\lambda(r)_{q-m}\}_{m=1}^{|S|}$.

We set a $(|S| \times |S|)$-matrix

$$A_2 = \begin{bmatrix} \binom{-Q_0}{1} & \cdots & \binom{-Q_0}{Q_0} & \binom{-Q_1}{1} & \cdots & \cdots & \binom{-Q_{T-1}}{M-q+1} \\ \binom{-Q_0}{2} & \cdots & \binom{-Q_0}{Q_0+1} & \binom{-Q_1}{2} & \cdots & \cdots & \binom{-Q_{T-1}}{M-q+2} \\ \vdots & & \vdots & \vdots & & & \vdots \\ \binom{-Q_0}{|S|} & \cdots & \binom{-Q_0}{Q_0+|S|-1} & \binom{-Q_1}{|S|} & \cdots & \cdots & \binom{-Q_{T-1}}{M-q+|S|} \end{bmatrix}.$$

Then $\det A_1 = \det A_2$ because

$$\begin{aligned} &\left(\alpha_m^{s,0}, \alpha_m^{s,1}, \ldots, \alpha_m^{s,M-q}\right) \\ &= \left(\binom{-Q_s}{m}, \binom{-Q_s}{m+1}, \ldots, \binom{-Q_s}{m+M-q}\right) \begin{bmatrix} 1 & \binom{Q_s}{1} & \binom{Q_s}{2} & \cdots & \binom{Q_s}{M-q} \\ 0 & 1 & \binom{Q_s}{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{Q_s}{2} \\ \vdots & & \ddots & \ddots & \binom{Q_s}{1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \end{aligned}$$

for $1 \leqslant s \leqslant T-1$ and a similar formula for $s=0$. So it sufficient to show that the matrix A_2 is non-singular. The following computation is used in a transformation of A_2. For $(s,k) \in S$, we obtain

$$\sum_{m=1}^{\infty} \sum_{j=0}^{m-1} \binom{Q_0}{m-1-j} \binom{-Q_s}{k+1+j} x^m$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{Q_0}{m-j} \binom{-Q_s}{k+1+j} x^{m+1} \\
&= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{Q_0}{j} \binom{-Q_s}{k+m+1-j} x^{m+1} \\
&= x^{-k} (1+x)^{Q_0} \left((1+x)^{-Q_s} - \sum_{j=0}^{k} \binom{-Q_s}{j} x^j \right) \\
&= x^{-k} (1+x)^{Q_0 - Q_s} - \sum_{t=0}^{Q_0} \sum_{j=0}^{k} \binom{Q_0}{t} \binom{-Q_s}{j} x^{t+j-k} \\
&= \sum_{m=0}^{\infty} \binom{Q_0 - Q_s}{m} x^{m-k} - \sum_{m=0}^{Q_0+k} \sum_{j=0}^{k} \binom{Q_0}{m-j} \binom{-Q_s}{j} x^{m-k} \\
&= \sum_{m=1}^{\infty} \binom{Q_0 - Q_s}{m+k} x^{m} - \sum_{m=1}^{Q_0} \sum_{j=0}^{k} \binom{Q_0}{k+m-j} \binom{-Q_s}{j} x^{m}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\binom{Q_0}{1} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\binom{Q_0}{|S|-1} & \cdots & \binom{Q_0}{1} & 1
\end{bmatrix}
\begin{bmatrix}
\binom{-Q_0}{1} & \binom{-Q_0}{2} & \cdots & \binom{-Q_0}{Q_0} \\
\binom{-Q_0}{2} & \binom{-Q_0}{3} & \cdots & \binom{-Q_0}{Q_0+1} \\
\vdots & \vdots & & \vdots \\
\binom{-Q_0}{|S|} & \binom{-Q_0}{|S|+1} & \cdots & \binom{-Q_0}{Q_0+|S|-1}
\end{bmatrix} \\
&= - \left[\begin{array}{cccc}
\binom{Q_0}{1} & \cdots & \binom{Q_0}{Q_0-1} & 1 \\
\vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\
\binom{Q_0}{Q_0-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\
1 & 0 & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots \\
0 & \cdots & \cdots & 0
\end{array}\right]
\begin{bmatrix}
1 & \binom{-Q_0}{1} & \cdots & \binom{-Q_0}{Q_0-1} \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \binom{-Q_0}{1} \\
0 & \cdots & 0 & 1
\end{bmatrix}
\end{aligned}
\tag{2.5}
$$

and for $1 \leqslant s \leqslant T-1$,

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\binom{Q_0}{1} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\binom{Q_0}{|S|-1} & \cdots & \binom{Q_0}{1} & 1
\end{bmatrix}
\begin{bmatrix}
\binom{-Q_s}{1} & \binom{-Q_s}{2} & \cdots & \binom{-Q_s}{M-p+1} \\
\binom{-Q_s}{2} & \binom{-Q_s}{3} & \cdots & \binom{-Q_s}{M-p+2} \\
\vdots & \vdots & & \vdots \\
\binom{-Q_s}{|S|} & \binom{-Q_s}{|S|+1} & \cdots & \binom{-Q_s}{M-p+|S|}
\end{bmatrix}
$$

$$= \left[\begin{array}{cccc} y_{10} & y_{11} & \cdots & y_{1,M-p} \\ \vdots & \vdots & & \vdots \\ y_{Q_0,0} & y_{Q_0,1} & \cdots & y_{Q_0,M-p} \\ \hline \binom{Q_0-Q_s}{Q_0+1} & \binom{Q_0-Q_s}{Q_0+2} & \cdots & \binom{Q_0-Q_s}{Q_0+M-p+1} \\ \binom{Q_0-Q_s}{Q_0+2} & \binom{Q_0-Q_s}{Q_0+3} & \cdots & \binom{Q_0-Q_s}{Q_0+M-p+2} \\ \vdots & \vdots & & \vdots \\ \binom{Q_0-Q_s}{|S|} & \binom{Q_0-Q_s}{|S|+1} & \cdots & \binom{Q_0-Q_s}{M-p+|S|} \end{array}\right] \tag{2.6}$$

where $y_{m,k}$ denotes $\binom{Q_0-Q_s}{m+k} - \sum_{j=0}^{k}\binom{Q_0}{m+k-j}\binom{-Q_s}{j}$ for $1 \leqslant m \leqslant Q_0$ and $0 \leqslant k \leqslant M-p$. We hence have $\det A_2 = (-1)^{Q_0(Q_0+1)/2} \det A_3$, where A_3 is the following $((|S|-Q_0) \times (|S|-Q_0))$-matrix:

$$A_3 = \begin{bmatrix} \binom{Q_0-Q_1}{Q_0+1} & \cdots & \binom{Q_0-Q_1}{Q_0+M-p+1} & \binom{Q_0-Q_2}{Q_0+1} & \cdots & \binom{Q_0-Q_{T-1}}{Q_0+M-p+1} \\ \binom{Q_0-Q_1}{Q_0+2} & \cdots & \binom{Q_0-Q_1}{Q_0+M-p+2} & \binom{Q_0-Q_2}{Q_0+2} & \cdots & \binom{Q_0-Q_{T-1}}{Q_0+M-p+2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \binom{Q_0-Q_1}{|S|} & \cdots & \binom{Q_0-Q_1}{M-p+|S|} & \binom{Q_0-Q_2}{|S|} & \cdots & \binom{Q_0-Q_{T-1}}{M-p+|S|} \end{bmatrix}. \tag{2.7}$$

It is proved in Appendix that A_3 is non-singular. □

Lemma 4. *Let V be simple and let $g, h \in G$ with $g \neq h$. Then $O_{g,n}(V) + O_{h,n}(V) = V$.*

Proof. For $g, h \in G$ with $g \neq h$, there are $k, r \in \{0, \ldots, T-1\}$ with $k \neq r$ and $v \in V$ such that $0 \neq v \in V^{(g,k)}$ and $v^{(h,r)} \neq 0$. For any $u \in V$ and $p \in \mathbb{Z}$, consider $F^{-r,p}(u, v, n)$ in Lemma 3. Then

$$F^{-r,p}(u,v,n) \equiv u_p^{(g,-r)} v \equiv 0 \quad \bigl(\mathrm{mod}\, O_{g,n}(V)\bigr)$$

from Lemma 1(1) because $u_p^{(g,-r)} v \in V^{(g,k-r)}$ and $k - r \not\equiv 0 \pmod{T}$. We also have

$$F^{-r,p}(u,v,n) \equiv u_p^{(h,-r)} v \equiv u_p^{(h,-r)} v^{(h,r)} \equiv u_p v^{(h,r)} \quad \bigl(\mathrm{mod}\, O_{h,n}(V)\bigr).$$

We hence have $u_p v^{(h,r)} \in O_{g,n}(V) + O_{h,n}(V)$ for any $u \in V$ and $p \in \mathbb{Z}$. Since $v^{(h,r)} \neq 0$ and V is simple, $V = \mathrm{Span}\{u_p v^{(h,r)} \mid u \in V,\ p \in \mathbb{Z}\}$ from Proposition 4.1 in [7]. So $O_{g,n}(V) + O_{h,n}(V) = V$ holds. □

Now we start to prove Theorem 1. Let $u, v \in V$ and fix $M \in \mathbb{Z}$ with $M \geqslant \mathrm{wt}(u) + \mathrm{wt}(v) - 1$. Define

$$u *_n v = \sum_{k=0}^{M+2l+1} \sum_{j=0}^{k} (-1)^{l-j} \binom{2l-j}{l} \binom{\mathrm{wt}(u)+l}{k-j} F^{-2l-1+k,0}(u,v,n).$$

We have $u *_n v \equiv u *_{g,n} v \pmod{O_{g,n}(V)}$ for any $g \in G$ from Lemma 3. If V is simple, then we have $A_{G,n}(V) \simeq \bigoplus_{g\in G} A_{g,n}(V)$ as algebras using the Chinese remainder theorem and Lemma 4. So (1) holds. (2) is clear from (1) and Lemma 1. □

3. A duality theorem of Schur–Weyl type

We always assume that V is simple throughout this section. A finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G,\mathcal{S})$ over $\mathbb{C}$ associated to G, a finite right G-set $\mathcal{S}$ and a suitable 2-cocycle α is constructed in [8]. $\mathcal{A}_\alpha(G,\mathcal{S})$ is called the generalized twisted double there. In this section we first review its construction in the case that $\mathcal{S}$ is a finite G-stable set of inequivalent irreducible twisted V-modules and a 2-cocycle α naturally determined by the G-action on $\mathcal{S}$. We will show a duality theorem of Schur–Weyl type for the actions of V^G and $\mathcal{A}_\alpha(G,\mathcal{S})$ on $\mathcal{M} = \bigoplus_{(g,M)\in\mathcal{S}} M$. That is, each simple $\mathcal{A}_\alpha(G,\mathcal{S})$ occurs in $\mathcal{M}$ and its multiplicity space is an irreducible V^G-module. Moreover, the different multiplicity spaces are inequivalent V^G-modules. It follows from this result that for any $g \in G$ every irreducible g-twisted module is a completely reducible V^G-module. These results are already shown in the case $g = 1$ in [8] and in the case where g is in the center of G in [11]. The basic ideas come from [3,8], and [11].

Let $\mathcal{T}_g$ be the set of all inequivalent irreducible g-twisted V-modules for $g \in G$ and set $\mathcal{T} = \{(g,M) \mid g \in G,\ M \in \mathcal{T}_g\}$. There is a natural right G-action on $\mathcal{T}$. Namely, for an irreducible g-twisted V-module (M, Y_M) and $a \in G$, we define

$$(M, Y_M)\cdot a = (M\cdot a, Y_{M\cdot a}).$$

Here $M \cdot a = M$ as a vector space and $Y_{M\cdot a}(u,z)$ is defined by

$$Y_{M\cdot a}(u,z) = Y_M(au,z) \quad \text{for } u \in V.$$

Note that $M\cdot a$ is an irreducible $a^{-1}ga$-twisted V-module. We set $(g,M)\cdot a = (a^{-1}ga, M\cdot a)$ for all $(g,M)\in\mathcal{T}$.

A subset $\mathcal{S} \subset \mathcal{T}$ is called *stable* if for any $(g,M) \in \mathcal{S}$ and $a \in G$ there exists $(a^{-1}ga, N) \in \mathcal{S}$ such that $M \circ a \simeq N$ as $a^{-1}ga$-twisted V-modules.

We assume that $\mathcal{S}$ is a finite G-stable subset of $\mathcal{T}$ until the end of this section. Let $(g,M)\in\mathcal{S}$ and $a \in G$. Then there exists $(aga^{-1}, N)\in\mathcal{S}$ such that $N\cdot a \simeq M$ as g-twisted V-modules. That is, there is an isomorphism $\phi(a,(g,M)) : M \to N$ of vector spaces such that

$$\phi\big(a,(g,M)\big)Y_{(g,M)}(u,z) = Y_{(aga^{-1},N)}(au,z)\phi\big(a,(g,M)\big)$$

for all $u \in V$. By the simplicity of M, there exists $\alpha_{(g,M)}(a,b)\in\mathbb{C}$ such that

$$\phi\big(a,\big(bgb^{-1}, M\cdot b^{-1}\big)\big)\phi\big(b,(g,M)\big) = \alpha_{(g,M)}(a,b)\phi\big(ab,(g,M)\big).$$

Moreover, for $a, b, c \in G$ and $(g, M) \in \mathcal{S}$ we have

$$\alpha_{(cgc^{-1}, M\cdot c^{-1})}(a, b)\alpha_{(g,M)}(ab, c) = \alpha_{(g,M)}(a, bc)\alpha_{(g,M)}(b, c).$$

Define a vector space $\mathbb{C}\mathcal{S} = \bigoplus_{(g,M)\in\mathcal{S}} \mathbb{C}e(g, M)$ with a basis $e(g, M)$ for $(g, M) \in \mathcal{S}$. The space $\mathbb{C}\mathcal{S}$ is an associative algebra under the product $e(g, M)e(h, N) = \delta_{(g,M),(h,N)}(h, N)$. Let $\mathcal{U}(\mathbb{C}\mathcal{S}) = \{\sum_{(g,M)\in\mathcal{S}} \lambda_{(g,M)}e(g, M) \mid \lambda_{(g,M)} \in \mathbb{C}^\times\}$ be the set of unit elements on $\mathbb{C}\mathcal{S}$ where $\mathbb{C}^\times$ is the multiplicative group of $\mathbb{C}$. $\mathcal{U}(\mathbb{C}\mathcal{S})$ is a multiplicative right G-module by the action $(\sum_{(g,M)\in\mathcal{S}} \lambda_{(g,M)}e(g, M)) \cdot a = \sum_{(g,M)\in\mathcal{S}} \lambda_{(g,M)}e(a^{-1}ga, M \cdot a)$ for $a \in G$. Set $\alpha(a, b) = \sum_{(g,M)\in\mathcal{S}} \alpha_{(g,M)}(a, b)e(g, M)$. Then

$$\bigl(\alpha(a, b) \cdot c\bigr)\alpha(ab, c) = \alpha(a, bc)\alpha(b, c)$$

holds for all $a, b, c \in G$. So $\alpha : G \times G \to \mathcal{U}(\mathbb{C}\mathcal{S})$ is a 2-cocycle.

Define the vector space $\mathcal{A}_\alpha(G, \mathcal{S}) = \mathbb{C}[G] \otimes \mathbb{C}\mathcal{S}$ with a basis $a \otimes e(g, M)$ for $a \in G$ and $(g, M) \in \mathcal{S}$ and a multiplication on it:

$$a \otimes e(g, M) \cdot b \otimes e(h, N) = \alpha_{(h,N)}(a, b)ab \otimes e\bigl((g, M) \cdot b\bigr)e(h, N).$$

Then $\mathcal{A}_\alpha(G, \mathcal{S})$ is an associative algebra with the identity element $\sum_{(g,M)\in\mathcal{S}} 1 \otimes e(g, M)$.

We define an action of $\mathcal{A}_\alpha(G, \mathcal{S})$ on $\mathcal{M} = \bigoplus_{(g,M)\in\mathcal{S}} M$ as follows: for $(g, M), (h, N) \in \mathcal{S},\ w \in N,\ a \in G$ we set

$$a \otimes e(g, M) \cdot w = \delta_{(g,M),(h,N)}\phi\bigl(a, (g, M)\bigr)w.$$

Note that the actions of $\mathcal{A}_\alpha(G, \mathcal{S})$ and V^G on $\mathcal{M}$ commute with each other.

For each $(g, M) \in \mathcal{S}$ set $G_{(g,M)} = \{a \in C_G(g) \mid M \cdot a \simeq M$ as g-twisted V-modules$\}$. Let $\mathcal{O}_{(g,M)}$ be the orbit of (g, M) under the action of G and let $G = \bigcup_{j=1}^{k} G_{(g,M)}g_j$ be a right coset decomposition with $g_1 = 1$. Then $\mathcal{O}_{(g,M)} = \{(g, M) \cdot g_j \mid j = 1, \dots, k\}$ and $G_{(g,M)\cdot g_j} = g_j^{-1}G_{(g,M)}g_j$. We define several subspaces of $\mathcal{A}_\alpha(G, \mathcal{S})$ by:

$$\begin{aligned}
S(g, M) &= \mathrm{Span}\bigl\{a \otimes e(g, M) \mid a \in G_{(g,M)}\bigr\},\\
D(g, M) &= \mathrm{Span}\bigl\{a \otimes e(g, M) \mid a \in G\bigr\} \quad \text{and}\\
D(\mathcal{O}_{(g,M)}) &= \mathrm{Span}\bigl\{a \otimes e(g, M) \cdot g_j \mid j = 1, \dots, k,\ a \in G\bigr\}.
\end{aligned}$$

Decompose $\mathcal{S}$ into a disjoint union of orbits

$$\mathcal{S} = \bigcup_{j\in J} \mathcal{O}_j.$$

Let (g_j, M^j) be a representative elements of $\mathcal{O}_j$. Then $\mathcal{O}_j = \{(g_j, M^j) \cdot a \mid a \in G\}$ and $\mathcal{A}_\alpha(G, \mathcal{S}) = \bigoplus_{j\in J} D(\mathcal{O}_{(g_j,M^j)})$. We recall the following properties of $\mathcal{A}_\alpha(G, \mathcal{S})$.

Lemma 5 [8, Lemma 3.4]. *Let $(g, M) \in \mathcal{S}$ and $G = \bigcup_{j=1}^{k} G_{(g,M)} g_j$. Then*

(1) *$S(g, M)$ is a subalgebra of $\mathcal{A}_\alpha(G, \mathcal{S})$ isomorphic to $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$, twisted group algebra with 2-cocycle $\alpha_{(g,M)}$.*
(2) *$D(\mathcal{O}_{(g,M)}) = \bigoplus_{j=1}^{k} D((g, M) \cdot g_j)$ is a direct sum of left ideals.*
(3) *Each $D(\mathcal{O}_{(g,M)})$ is a two sided ideal of $\mathcal{A}_\alpha(G, \mathcal{S}) = \bigoplus_{j \in J} D(\mathcal{O}_{(g_j,M^j)})$. Moreover, $D(\mathcal{O}_{(g,M)})$ has the identity element $\sum_{(h,N)\in\mathcal{O}_{(g,M)}} 1 \otimes e(h, N)$.*

Lemma 6 [8, Theorem 3.6].

(1) *$D(\mathcal{O}_{(g,M)})$ is semisimple for all $(g, M) \in \mathcal{S}$ and the simple $D(\mathcal{O}_{(g,M)})$-modules are precisely equal to $\mathrm{Ind}_{S(g,M)}^{D(g,M)} W = D(g, M) \otimes_{S(g,M)} W$ where W ranges over the simple $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$-modules.*
(2) *$\mathcal{A}_\alpha(G, \mathcal{S})$ is semisimple and simple $\mathcal{A}_\alpha(G, \mathcal{S})$-modules are precisely $\mathrm{Ind}_{S(g_j,M^j)}^{D(g_j,M^j)} W$ where W ranges over the simple $\mathbb{C}^{\alpha_{(g_j,M^j)}}[G_{(g_j,M^j)}]$-modules and $j \in J$.*

Let $(g, M) \in \mathcal{T}$. Following [12] we define weight zero operator $o_M(u)$ by $u_{\mathrm{wt}(u)-1}$ on M for homogeneous $u \in V$ and extend $o_M(u)$ to all u by linearity. For any nonnegative rational number $n \in \mathbb{Z}/T$, define a map σ_n from $A_{G,n}(V)$ to $\bigoplus_{(g,M)\in\mathcal{S}} \bigoplus_{\substack{m\in\mathbb{Z}/T\\ 0\leqslant m\leqslant n}} \mathrm{End}\, M(m)$ by

$$\sigma_n(u) = \sum_{(g,M)\in\mathcal{S}} \sum_{\substack{m\in\mathbb{Z}/T\\ 0\leqslant m\leqslant n}} o_M(u).$$

For $a \in G$ and $f \in \mathrm{End}(M)$, define the action of a by

$$a \cdot f = \phi\big(a, (g, M)\big) f \phi\big(a, (g, M)\big)^{-1} \in \mathrm{End}\big(M \cdot a^{-1}\big).$$

This defines a left action of G on $\bigoplus_{(g,M)\in\mathcal{S}} \bigoplus_{\substack{m\in\mathbb{Z}/T\\ 0\leqslant m\leqslant n}} \mathrm{End}\, M(m)$.

We prepare the following results in order to show the main result in this section. For any $(g, M) \in \mathcal{S}$, we always arrange the grading on $M = \bigoplus_{0\leqslant j\in\mathbb{Z}/T} M(j)$ so that $M(0) \neq 0$ if $M \neq 0$ using a grading shift.

Lemma 7. *Let $(g, M), (h, N) \in \mathcal{S}$. Let $m \in \mathbb{Z}/|g|, n \in \mathbb{Z}/|h|$ such that $m \leqslant n$. If $M(m) \neq 0$ and $M(m) \simeq N(n)$ as $A_{G,n}(V)$-modules, then $(g, M) = (h, N)$ and $m = n$.*

Proof. Since $M(m) \neq 0$ and $M(m) \simeq N(n)$ as $A_{G,n}(V)$-modules, we have $g = h$ from Theorem 1. So $M(m) \simeq N(n)$ as $A_{g,n}(V)$-modules. We have $m = n$ and $M = N$ by Theorem 4.3 in [6]. □

Lemma 8. *The map σ_n is a G-module epimorphism. In particular,*

$$\sigma_n\bigl(A_{G,n}(V)^G\bigr) = \left(\bigoplus_{(g,M)\in\mathcal{S}} \bigoplus_{\substack{m\in\mathbb{Z}/T\\ 0\leqslant m\leqslant n}} \operatorname{End} M(m)\right)^G.$$

Proof. The proof is similar to that of in Lemma 6.13 in [8] because of Lemma 7. □

For $(g,M)\in\mathcal{S}$ let $\Lambda_{G_{(g,M)},\alpha_{(g,M)}}$ be the set of all irreducible characters λ of $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$. We denote the corresponding simple module by $W(g,M)_\lambda$. Note that M is a semisimple $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$-module. Let M^λ be the sum of simple $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$-module of M isomorphic to $W(g,M)_\lambda$. Then

$$M = \bigoplus_{\lambda\in\Lambda_{G_{(g,M)},\alpha_{(g,M)}}} M^\lambda.$$

Moreover $M^\lambda = W(g,M)_\lambda\otimes M_\lambda$ where $M_\lambda = \operatorname{Hom}_{\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]}(W(g,M)_\lambda, M)$ is the multiplicity of $W(g,M)_\lambda$ in M. We can realize M_λ as a subspace of M in the following way: Let $w\in W(g,M)_\lambda$ be a fixed nonzero vector. Then we can identify $\operatorname{Hom}_{\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]}(W(g,M)_\lambda, M)$ with the subspace

$$\bigl\{f(w)\mid f\in \operatorname{Hom}_{\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]}\bigl(W(g,M)_\lambda, M\bigr)\bigr\},$$

of M^λ. Note that the actions of $\mathbb{C}^{\alpha_{(g,M)}}[G_{(g,M)}]$ and $V^{G_{(g,M)}}$ on M commute with each other. So M^λ and M_λ are ordinary $V^{G_{(g,M)}}$-modules. Furthermore, M^λ and M_λ are ordinary V^G-modules. It is shown by Theorem 3.9 in [11] that M_λ is a nonzero irreducible $V^{G_{(g,M)}}$-module for any $\lambda\in\Lambda_{G_{(g,M)},\alpha_{(g,M)}}$.

Let $\mathcal{S}=\bigcup_{j\in J}\mathcal{O}_j$ be an orbit decomposition and fix $(g_j,M^j)\in\mathcal{O}_j$ for each $j\in J$. For convenience, we set

$$G_j = G_{(g_j,M^j)},\quad \Lambda_j = \Lambda_{(g_j,M^j),\alpha_{(g_j,M^j)}}\quad\text{and}\quad W_{j,\lambda} = W\bigl(g_j,M^j\bigr)_\lambda$$

for $j\in J$ and $\lambda\in\Lambda_j$. We have a decomposition

$$M^j = \bigoplus_{\lambda\in\Lambda_j} W_{j,\lambda}\otimes M^j_\lambda,$$

as a $\mathbb{C}^{\alpha_{(g_j,M^j)}}[G_j]\otimes V^{G_j}$-module. We also have

$$\mathcal{M} = \bigoplus_{j\in J,\ \lambda\in\Lambda_j} \operatorname{Ind}_{S(g_j,M^j)}^{D(g_j,M^j)} W_{j,\lambda}\otimes M^j_\lambda$$

as a $\mathcal{A}_\alpha(G,\mathcal{S})\otimes V^G$-module using the same arguments as those in Proposition 6.5 in [8].

For $j \in J$ and $\lambda \in \Lambda_j$ we set

$$W_\lambda^j = \mathrm{Ind}_{S(g_j, M^j)}^{D(g_j, M^j)} W_{j,\lambda}.$$

Then W_λ^j forms a complete list of simple $\mathcal{A}_\alpha(G, \mathcal{S})$-modules from Lemma 6. Now we have the main result in this section.

Theorem 2. *As a $\mathcal{A}_\alpha(G, \mathcal{S}) \otimes V^G$-module,*

$$\mathcal{M} = \bigoplus_{j \in J,\ \lambda \in \Lambda_j} W_\lambda^j \otimes M_\lambda^j.$$

Moreover

(1) *Each M_λ^j is a nonzero irreducible V^G-module.*
(2) *$M_{\lambda_1}^{j_1}$ and $M_{\lambda_2}^{j_2}$ are isomorphic V^G-module if and only if $j_1 = j_2$ and $\lambda_1 = \lambda_2$.*

In particular, all irreducible g-twisted V-modules are completely irreducible V^G-modules.

Proof. The proof is similar to that of Theorem 6.14 in [8] because of Lemma 8. □

Acknowledgment

We thank Soichi Okada for information of a method to compute the determinant of the matrix in Appendix.

Appendix

In this appendix we prove the matrix A_3 in (2.7) is non-singular. Let a, b, t positive integers and $x_1, x_2, \dots, x_t$ indeterminants. Set a $(bt \times bt)$-matrix

$$A = \begin{bmatrix} \binom{x_1}{a} & \binom{x_1}{a+1} & \cdots & \binom{x_1}{a+b-1} & \binom{x_2}{a} & \cdots & \binom{x_2}{a+b-1} & \cdots & \binom{x_t}{a+b-1} \\ \binom{x_1}{a+1} & \binom{x_1}{a+2} & \cdots & \binom{x_1}{a+b} & \binom{x_2}{a+1} & \cdots & \binom{x_2}{a+b} & \cdots & \binom{x_t}{a+b} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \binom{x_1}{a+bt-1} & \binom{x_1}{a+bt} & \cdots & \binom{x_1}{a+bt+b-2} & \binom{x_2}{a+bt-1} & \cdots & \binom{x_2}{a+bt+b-2} & \cdots & \binom{x_t}{a+bt+b-2} \end{bmatrix}.$$

It is proved from the following result that the matrix A_3 is non-singular.

Proposition 9. *Let*

$$H(x_1,\dots,x_t)=\prod_{i=1}^{t}\left(\prod_{j=1}^{b-1}(x_i+j)^{b-j}\prod_{j=0}^{a-1}(x_i-j)^{b}\prod_{j=1}^{b-1}(x_i-a+1-j)^{b-j}\right)$$
$$\times\prod_{1\leqslant i<j\leqslant t}\ \prod_{k=-b+1}^{b-1}(x_i-x_j+k)^{b-|k|}.$$

Then

$$\det A=(-1)^{bt(bt-1)/2}\frac{H(x_1,\dots,x_t)}{H(a+tb-1,a+(t-1)b-1,\dots,a+b-1)}.$$

Proof. Since $\binom{x}{j}=x(x-1)\cdots(x-j+1)/j!$, $\prod_{j=0}^{a-1}(x_i-j)^b\prod_{j=1}^{b-1}(x_i-a+1-j)^{b-j}$ is a factor of $\det A$ for any $1\leqslant i\leqslant t$. For any $1\leqslant i\leqslant t$ and $0\leqslant k\leqslant b-1$, set a $(bt\times b)$-matrix $B(i,k)$ by

$$B(i,k)=\begin{bmatrix}\binom{x_i}{a} & \binom{x_i+1}{a+1} & \cdots & \binom{x_i+k}{a+k} & \binom{x_i+k}{a+k+1} & \cdots & \binom{x_i+k}{a+b-1}\\ \binom{x_i}{a+1} & \binom{x_i+1}{a+2} & \cdots & \binom{x_i+k}{a+k+1} & \binom{x_i+k}{a+k+2} & \cdots & \binom{x_i+k}{a+b}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \binom{x_i}{a+bt-1} & \binom{x_i+1}{a+bt} & \cdots & \binom{x_i+k}{a+bt-1+k} & \binom{x_i+k}{a+bt+k} & \cdots & \binom{x_i+k}{a+bt+b-2}\end{bmatrix}.$$

For any $1\leqslant k\leqslant b-1$, $\det A$ is equal to that of the $(bt\times bt)$-matrix $[B(1,k)\ B(2,k)\cdots B(t,k)]$ because $\binom{x}{j}+\binom{x}{j+1}=\binom{x+1}{j+1}$. So $(x_i+k)^{b-k}$ is a factor of $\det A$ for any $1\leqslant i\leqslant t$ and $1\leqslant k\leqslant b-1$. Fix any $1\leqslant i<j\leqslant t$ and $0\leqslant k\leqslant b-1$. Then $\det A$ is equal to that of the $(bt\times bt)$-matrix

$$\left[B(1,0)\cdots B(i,0)\cdots B(j,k)\cdots B(t,0)\right]$$

by the same reason as above. Comparing the $(t(i-1)+p)$th and $(t(j-1)+p)$th columns for all $k+1\leqslant p\leqslant b$, we have $(x_i-x_j-k)^{b-k}$ is a factor of $\det A$. We also have that $(x_i-x_j+k)^{b-k}$ is a factor of $\det A$ by applying the same argument to the matrix

$$\left[B(1,0)\cdots B(i,k)\cdots B(j,0)\cdots B(t,0)\right].$$

So there is $\alpha(a,b)\in\mathbb{C}[x_1,\dots,x_t]$ such that

$$\det A=\alpha(a,b)H(x_1,\dots,x_n). \tag{A.1}$$

We have $\alpha(a,b)\in\mathbb{C}$ since the degrees of both sides in formula (A.1) are equal to $(a-1)bt+b^2t(t+1)/2$. Substituting $(a+tb-1,a+(t-1)b-1,\dots,a+b-1)$ for $(x_1,x_2,\dots,x_t)$ in A, we have an anti-diagonal matrix with all anti-diagonal elements 1. Hence $\alpha(a,b)=(-1)^{bt(bt-1)/2}/H(a+tb-1,a+(t-1)b-1,\dots,a+b-1)$. $\square$

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